

LPTENS 00/22  
AEI-2000-032

# Currents and Superpotentials in classical gauge theories: II. Global aspects and the example of Affine gravity

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## ABSTRACT

The conserved charges associated to gauge symmetries are defined at a boundary component of space-time because the corresponding Noether current can be rewritten on-shell as the divergence of a superpotential. However, the latter is afflicted by ambiguities. Regge and Teitelboim found a procedure to lift the arbitrariness in the Hamiltonian framework. An alternative covariant formula was proposed by one of us for an arbitrary variation of the superpotential, it depends only on the equations of motion and on the gauge symmetry under consideration. Here we emphasize that in order to compute the charges, it is enough to stay at a boundary of spacetime, without requiring any hypothesis about the bulk or about other boundary components, so one may speak of holographic charges. It is well known that the asymptotic symmetries that lead to conserved charges are really defined at infinity, but the choice of boundary conditions and surface terms in the action and in the charges is usually determined through integration by parts whereas each component of the boundary should be considered separately. We treat the example of gravity (for any space-time dimension, with or without cosmological constant), formulated as an Affine theory which is a natural generalization of the Palatini and Cartan-Weyl (vielbein) first order formulations. We then show that the superpotential associated to a Dirichlet boundary condition on the metric (the one needed to treat asymptotically flat or AdS spacetimes) is the one proposed by Katz, Bičák and Lynden-Bell and not that of Komar. We finally discuss the KBL superpotential at null infinity.

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<sup>1</sup>UMR 8549 du CNRS et de l'École Normale Supérieure. This work has been partly supported by the EU TMR contract ERBFMRXCT96-0012.

# 1 Introduction

Any Noether current associated to a gauge symmetry can be rewritten on-shell as the divergence of a superpotential. This is a very general property and does not depend on the gauge invariant theory we are considering [1]. Let us be more precise. Suppose that a gauge symmetry variation is given locally by

$$\delta_\xi \varphi^i = d\xi^\alpha \wedge \Delta_\alpha^i + \xi^\alpha \wedge \tilde{\Delta}_\alpha^i \quad (1)$$

where  $\varphi^i$  denotes a  $p_i$ -form field. The  $(p_i - 1)$  and  $p_i$  space-time forms,  $\Delta_\alpha^i$  and  $\tilde{\Delta}_\alpha^i$ , are functions of the fields. Then the Noether current associated to a one parameter symmetry subgroup (1) is *always* given by:

$$\mathbf{J}_\xi = d\mathbf{U}_\xi + \mathbf{W}_\xi \quad (2)$$

with the definition

$$\mathbf{W}_\xi := \xi^\alpha \Delta_\alpha^i \wedge \mathbf{E}_i, \quad (3)$$

where the  $(D - p_i)$ -forms  $\mathbf{E}_i := \frac{\delta \mathbf{L}}{\delta \varphi^i}$  are the Euler-Lagrange equations associated to the field  $\varphi^i$ . We used equation (1) in the definition (3). Note that the *Noether equation*  $d\mathbf{J}_\xi = \delta_\xi \varphi^i \wedge \mathbf{E}_i$  follows from (2) and from the *Noether identities* due to the gauge symmetry, namely,  $d\mathbf{W}_\xi = \delta_\xi \varphi^i \wedge \mathbf{E}_i$ . The current is associated to a one-parameter subgroup of “rigid” symmetries when  $\xi^\alpha$  can be globally taken as constant or at least has a canonical spacetime dependence as in the case of rotations, but the form (2) follows from local gauge invariance along that one parameter subgroup.

In our previous work [1], we emphasized the fact that the Noether method does not define the current  $\mathbf{J}_\xi$  and its superpotential  $\mathbf{U}_\xi$  unambiguously. This is the well-known fact that the Noether current is defined up to some exact form, namely  $\mathbf{J}_\xi \sim \mathbf{J}_\xi + d\mathbf{Y}$ . This exact term may contribute to the Noether charge when the space-time has some boundary with non-vanishing fields on it. A *case by case* prescription which depends on the boundary conditions is then needed in order to define  $\mathbf{J}_\xi$ . An attempt to give a general formula for  $\mathbf{U}_\xi$  can give rise to incorrect results; see for instance our remarks in the case of supergravities [2]. The same problem arises in the covariant symplectic phase space formalism. In that case, the ambiguity on the symplectic form can be fixed by some “covariant” criterion as in [3].

In [4], a way to solve the ambiguity in the Noether current was proposed. The main result was to give a formula for an *arbitrary* variation of the

superpotential, namely<sup>2</sup>:

$$\int_{\mathcal{B}_r} \delta \mathbf{U}_\xi = - \int_{\mathcal{B}_r} \delta \varphi^i \wedge \frac{\partial \mathbf{W}_\xi}{\partial d\varphi^i} \quad (4)$$

We assume that our spacetime  $\mathcal{M}$  is bounded by a set of  $n$   $(D - 1)$ -dimensional time-like (or null) hypersurfaces denoted by  $\mathcal{H}_r$ ,  $r = \{1, \dots, n\}$ . Then,  $r$  labels the connected time-like (or null) components of  $\partial\mathcal{M}$ . We denote by  $\Sigma_t$  a space-like Cauchy hypersurface at fixed time  $t$ . Then, the closed  $(D - 2)$ -dimensional manifold  $\mathcal{B}_r$  of (4) is defined by  $\mathcal{B}_r = \Sigma_t \cap \mathcal{H}_r$ , for some  $r = \{1, \dots, n\}$  (and then  $\partial\Sigma_t = \sum_{r=1}^n \mathcal{B}_r$ ). In our gravitational example, we will choose  $\mathcal{B}_r$  to be spatial infinity at fixed time  $\mathcal{B}_\infty$ .

It is important to recall that the formula (4) holds only if the theory has been rewritten in a *first order formalism*<sup>3</sup>, in the sense that both, the equations of motion and the symmetries (1), depend at most on the first derivatives of the fields [4]. Note that this definition *does not imply* that a 1<sup>st</sup> order Lagrangian is *linear* in the first derivatives of the fields. The simplest example of a 1<sup>st</sup> order theory *quadratic* in the first derivatives of the fields is the 5-dimensional (abelian) Chern-Simons Lagrangian  $\mathbf{L}_{CS} = \mathbf{A} \wedge d\mathbf{A} \wedge d\mathbf{A}$ .

The formula for the superpotential (4) is non-ambiguous because it depends only on the equations of motion and on the functional form of the gauge symmetry (1) of the theory. It expresses an arbitrary variation of the superpotential to be integrated on  $\mathcal{B}_r$  (for any  $r = \{1, \dots, n\}$ ) taking into account the chosen boundary conditions. No additional information on the behavior of the fields outside of  $\mathcal{H}_r$  is needed for *consistency* [5]. In fact, equation (4) can be justified using only the value of the fields (and their derivatives) at  $\mathcal{H}_r$ , no matter what happens in the bulk or on the other boundaries  $\mathcal{H}_s$ ,  $s \neq r$ . This is indeed expected from our experience: The ADM mass gives the total mass at spatial infinity, independently of the number of black holes inside the spacetime (and then independently of the boundary conditions used to describe their horizons). This has to be contrasted with the Hamiltonian Regge and Teitelboim procedure [7]. There, the requirement of *differentiability* of the generators of first class constraints implies that the Hamiltonian version of equation (4) has to be satisfied on *every*  $\mathcal{B}_r$ . The same stronger condition is also needed in the so-called co-variant phase space symplectic formalism for consistency (see for example [8, 9, 3]).

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<sup>2</sup>Additional motivation for the use of (4) is given in a companion paper [5].

<sup>3</sup>About the possibility to construct a first order theory from a second order one preserving the symmetries, see [6].

The purpose of this paper is to use formula (4) to compute the superpotentials associated to general relativity<sup>4</sup>, with or without cosmological constant, in *any* spacetime dimension ( $D \geq 3$ ). As stated already in [1] (using there a case by case prescription) we find that the superpotential associated to a “Dirichlet” boundary condition on the metric is the one proposed by Katz, Bičák and Lynden-Bell [11]. The superpotential associated to Dirichlet boundary conditions on the connection (ie Neumann condition on the metric), is *one half* (in four spacetime dimensions) the famous Komar [12] superpotential<sup>5</sup>. We use here the  $gl(D, \mathbb{R})$  first order formalism developed in [1]. The motivation is that the *Affine- $gl(D, \mathbb{R})$*  formalism generalizes both the Palatini (dear to relativists) and the tetrad-vielbein (needed for supergravities) formalisms. So a computation at the  $gl(D, \mathbb{R})$ -level can be pulled back to either of these two well-known formalisms without much additional work.

We would like to insist on the simplicity and on the absence of any ambiguity or additional criteria of our present derivation of these gravitational superpotentials. In particular, the (general) covariance of our results is automatic and does not have to be required by hand.

We finally comment on the use of the KBL superpotential at null infinity.

## 2 The $gl(D, \mathbb{R})$ formalism for gravity and the associated superpotentials

### 2.1 The $gl(D, \mathbb{R})$ gravity

The basic idea of the  $gl(D, \mathbb{R})$  formalism is to unify the two known first order formulations of General Relativity, namely the Palatini and the vielbein (orthonormal frames) formulations, both will follow from partial extremisations of our action. Let us recall the results of [1].

In the Palatini case, the *torsionless condition* for the connection (namely  $\Gamma_{\nu\rho}^{\mu} = \Gamma_{\rho\nu}^{\mu}$ ) is *assumed* from the beginning. The *metric compatibility* of the connection with the metric ( $\nabla_{\mu}g_{\nu\rho} = 0$ ) and the Einstein equations are derived from the variational principle.

On the other hand, the vielbein formulation *assumes* the *metric compatibility condition* between the flat Minkowski metric and the associated or-

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<sup>4</sup>The examples of Yang-Mills, higher dimensional Chern-Simons and supergravity theories were respectively studied in [4] and [2] (see also [10] for a review).

<sup>5</sup>This result was rediscovered in [13].

thonormal connection, that is,  $D_\mu \eta^{ab} = \partial_\mu \eta^{ab} + \omega_{\mu c}^a \eta^{cb} + \omega_{\mu c}^b \eta^{ac} = 2\omega_\mu^{(ab)} = 0$ . The *torsionless condition* ( $D_{[\mu} \theta_{\nu]}^a = 0$ ) and Einstein equations follow from the equations of motion of the basic fields.

The  $gl(D, \mathbb{R})$  first order formulation combines both ideas in a nice way: nothing is assumed from the beginning and the *metric compatibility* and *torsionless condition* are derived from the equations of motion of the connection of the linear frame bundle  $gl(D, \mathbb{R})$  (after fixing the Projective symmetry in the Einstein gauge). The Einstein equations are recovered as usual.

## The Lagrangian and the equations of motion

The Lagrangian of the  $gl(D, \mathbb{R})$  gravity is a D-form  $L$  ( $D \geq 3$  is the space-time dimension), it is a function of a linear 1-form connection  $\omega_b^a$  (for a Yang-Mills type  $gl(D, \mathbb{R})$ ), of the canonical 1-form  $\theta^a$  ( $\mathbb{R}^D$  valued) and of the metric  $g^{ab}$  (which will be used to lift and lower the  $\mathbb{R}^D$ -valued indices), as well as their first derivatives:

$$L = -\frac{1}{4\kappa^2} R_c^a \wedge \sqrt{|g|} g^{cb} \Sigma_{ab} \quad (5)$$

Where  $4\kappa^2 = 16\pi G$  and

$$\Sigma_{a_1 \dots a_r} := \frac{1}{(D-r)!} \epsilon_{a_1 \dots a_r c_{r+1} \dots c_D} \theta^{c_{r+1}} \wedge \dots \wedge \theta^{c_D} \quad (6)$$

$\epsilon_{a_1 \dots a_D}$  being the Levi-Civita symbol ( $\epsilon_{0 \dots (D-1)} = 1$ ).

Each field of the theory has a curvature,

$$R_b^a := d\omega_b^a + \omega_c^a \wedge \omega_b^c \quad (7)$$

$$\Theta^a := D\theta^a = d\theta^a + \omega_b^a \wedge \theta^b \quad (8)$$

$$\Xi^{ab} := Dg^{ab} = dg^{ab} + \omega^{ab} + \omega^{ba} \quad (9)$$

called respectively the curvature 2-form, the torsion and the nonmetricity.

The Euler-Lagrange equations corresponding to (5) are given by:

$$\frac{\delta L}{\delta g^{ab}} = -\frac{\sqrt{|g|}}{8\kappa^2} \left( R_c^a \wedge \Sigma_{cb} + R_b^c \wedge \Sigma_{ca} - g_{ab} R^{cd} \wedge \Sigma_{cd} \right) \quad (10)$$

$$\frac{\delta L}{\delta \theta^a} = -\frac{\sqrt{|g|}}{4\kappa^2} R^{bc} \wedge \Sigma_{bca} \quad (11)$$

$$\frac{\delta L}{\delta \omega_b^a} = -\frac{1}{4\kappa^2} D \left( \sqrt{|g|} g^{bc} \Sigma_{ac} \right) = -\frac{\sqrt{|g|}}{4\kappa^2} \left( \Xi^{bc} \wedge \Sigma_{ac} + \Theta^c \wedge \Sigma_{aec} g^{eb} \right) \quad (12)$$

with  $\mathcal{E}^{ab} := \Xi^{ab} - g^{ab} \frac{\Xi}{2}$  and  $\Xi := \Xi^{ab} g_{ab}$ .

### The gauge symmetries

The Lagrangian (5) is invariant under three gauge symmetries:

1) *The local “frame choice” freedom*, parametrized by an arbitrary infinitesimal local matrix of  $gl(D, \mathbb{R})$  namely,  $\lambda^a_b = \lambda^a_b(x)$ . The variations of the fields are:

$$\begin{aligned}\delta_\lambda \theta^a &= \lambda^a_b \theta^b \\ \delta_\lambda g^{ab} &= \lambda^a_c g^{cb} + \lambda^b_c g^{ac} \\ \delta_\lambda \omega^a_b &= -D\lambda^a_b = -d\lambda^a_b - \omega^a_c \lambda^c_b + \omega^c_b \lambda^a_c\end{aligned}\quad (13)$$

2) *The diffeomorphism invariance*, parametrized by an arbitrary infinitesimal vector field  $\xi^\rho = \xi^\rho(x)$ :

$$\delta_\xi \theta^a = \mathcal{L}_\xi \theta^a \quad (14)$$

and so on for  $g^{ab}$  and  $\omega^a_b$ . Here  $\mathcal{L}_\xi$  is the usual Lie derivative acting on forms and is given by  $\mathcal{L}_\xi = d \cdot \mathbf{i}_\xi + \mathbf{i}_\xi \cdot d$ .

3) *The Projective Symmetry*, parametrized by an arbitrary infinitesimal one-form  $\kappa = \kappa(x)$ :

$$\begin{aligned}\delta_\kappa \theta^a &= \delta_\kappa g^{ab} = 0 \\ \delta_\kappa \omega^a_b &= \kappa \delta^a_b\end{aligned}\quad (15)$$

### Palatini and vielbein formalisms

As was shown in [1], the new feature here is that we need to fix the Projective symmetry (15) in what we called the Einstein gauge to recover the torsionless and metricity conditions from the equations of motion of the linear connection (12). The physics, namely Einstein equations, does not depend on this gauge choice.

Now, the Palatini formalism can be recovered after fixing *all* the  $gl(D, \mathbb{R})$  symmetry by the canonical choice  $\theta_\mu^a = \delta_\mu^a$ . On the other hand, the Cartan-Weyl (vielbein or orthonormal frames) formalism comes after choosing  $\theta_\mu^a =$

$e_\mu^a, e_\mu^a$  being an orthonormal frame (i.e.  $g^{ab} = \eta^{ab}$ , with  $\eta^{ab}$  the ordinary flat Minkowski metric). This last choice breaks the local  $gl(D, \mathbb{R})$  down to local  $so(D, \mathbb{R})$ .

The above gauge fixings can be generally rewritten as  $\theta_\mu^a = \bar{\theta}_\mu^a$ , with  $\bar{\theta}_\mu^a(x)$  being an arbitrary given frame. The *residual* gauge symmetry which preserves such a choice is a linear combination of a diffeomorphism and a  $gl(D, \mathbb{R})$  rotation such that:

$$\mathcal{L}_\xi \bar{\theta}^a + \delta_\lambda \bar{\theta}^a = 0 \quad (16)$$

The above equation gives the parameter  $\lambda_b^a$  in terms of the diffeomorphism parameter  $\xi^\rho$ . Using now the identity  $\mathcal{L}_\xi \bar{\theta}^a = D\xi^a + i_\xi \bar{\Theta}^a - i_\xi \omega_b^a \bar{\theta}^b$  (where  $\xi^a := \xi^\rho \theta_\rho^a$  and  $\bar{\Theta}^a$  is the torsion (8) with  $\theta^a = \bar{\theta}^a$ ) and the torsionless condition, equation (16) becomes:

$$(i_\xi \omega_b^a - \lambda_b^a) \bar{\theta}^b = D\xi^a \quad (17)$$

In conclusion, after fixing the  $gl(D, \mathbb{R})$  gauge symmetry, the remaining symmetry is diffeomorphisms parametrized by  $\xi^\rho$  with simultaneous  $gl(D, \mathbb{R})$ -rotations parametrized by  $\lambda_b^a[\xi]$  which satisfy (17) (or equivalently (16)).

Let us clarify the above point by a simple example: in the Palatini case ( $\theta_\mu^a = \bar{\theta}_\mu^a = \delta_\mu^a$ ), equation (16) (or (17)) simply gives  $\lambda_b^a = -\partial_b \xi^a$  (see also [1]). Using equations (13-14), we then check that the *residual* symmetry is the usual diffeomorphism symmetry:

$$\begin{aligned} \delta_{\xi, \lambda} g^{ab} &= \mathcal{L}_\xi g^{ab} + \lambda_c^a g^{cb} + \lambda_b^c g^{ac} \\ &= \xi^c \partial_c g^{ab} - \partial_c \xi^a g^{cb} - \partial_c \xi^b g^{ac} \\ \delta_{\xi, \lambda} \omega_b^a &= \mathcal{L}_\xi \omega_b^a - d\lambda_b^a - \omega_d^a \lambda_b^d + \omega_b^d \lambda_d^a \\ &= (\xi^d \partial_d \Gamma_{cb}^a + \partial_c \xi^d \Gamma_{db}^a + \partial_c \partial_b \xi^a + \Gamma_{cd}^a \partial_b \xi^d - \Gamma_{cb}^d \partial_d \xi^a) dx^c \end{aligned}$$

with  $\omega_b^a := \Gamma_{cb}^a dx^c$ .

## The Dirichlet and Neumann boundary conditions

The stationarity of the action constructed from the Hilbert Lagrangian (5) implies the following condition at spatial infinity<sup>6</sup>:

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<sup>6</sup>For consistency, the variational principle is required to be satisfied at least near the boundary component where the superpotential and the conserved charges are computed [4, 5].

$$-\int_{\Sigma_\infty} \delta\omega^a_b \wedge \Sigma_a^b = 0 \quad (18)$$

where we used the compact notation:

$$\Sigma_a^b := \frac{\sqrt{|g|}}{4\kappa^2} g^{bc} \Sigma_{ac} \quad (19)$$

It is possible to add an appropriate total derivative  $dK$  to the Lagrangian (5) in order to implement a given boundary condition and solve (18). The two generic cases are the following:

$$D : \quad -\int_{\Sigma_\infty} \delta\omega^a_b \wedge \Delta\Sigma_a^b = 0 \quad (20)$$

$$N : \quad \int_{\Sigma_\infty} \Delta\omega^a_b \wedge \delta\Sigma_a^b = 0 \quad (21)$$

and the definitions,

$$\Delta\Sigma_a^b := \Sigma_a^b - \bar{\Sigma}_a^b, \quad \Delta\omega^a_b := \omega^a_b - \bar{\omega}^a_b \quad (22)$$

with  $\bar{\Sigma}_a^b$  and  $\bar{\omega}^a_b$  our chosen asymptotic background fields<sup>7</sup> whose arbitrary variation vanishes,  $\delta\bar{\Sigma}_a^b = \delta\bar{\omega}^a_b = 0$ . Then, the equations (20-21) are obtained by adding respectively to the Lagrangian (5) the following surface terms  $-d(\omega^a_b \wedge \bar{\Sigma}_a^b)$  and  $d(\Delta\omega^a_b \wedge \Sigma_a^b)$ .

We then recognize from (20-21) typical Dirichlet and Neumann boundary conditions: we respectively fix the asymptotic value of the metric (through  $\bar{\Sigma}_a^b$ ) or of the connection (through  $\bar{\omega}^a_b$ ). Any linear combination of (20-21) is also an appropriate boundary condition, compatible with a variational principle.

We will be mostly interested in the cases where the metric is asymptotically flat (or AdS) at spatial infinity. In that case, the appropriate boundary condition is given by (20), with  $\bar{\Sigma}_a^b$  constructed from the flat (or AdS) metric. A covariant way to check the vanishing of (20) is to use the compactification of Ashtekar and Romano [14] for spatial infinity. For an asymptotically AdS space, we can use the usual compactification of Penrose [15] (see also the recent [16]). We will not give the complete proof of these

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<sup>7</sup>The quantity  $\bar{\Sigma}_a^b$  is constructed with some chosen asymptotic background metric. For example, in the Palatini formalism, we use  $\theta_\mu^a = \delta_\mu^a$  and  $g^{ab} = \bar{g}^{ab}$  in definition (19). In the vielbein formalism, we fix at infinity the orthonormal frame by  $\theta_\mu^a = \bar{e}_\mu^a$  in the gauge  $g^{ab} = \eta^{ab}$  (see above or [1]).



statements here. The case of null infinity is quite a bit more involved since neither (20) nor (21) are satisfied there [17]. We shall elaborate on this in section 3 (and in [5]).

## 2.2 The associated superpotentials

The superpotentials associated to the gauge symmetries (13-15) have been computed with the cascade equations in [1]. The background contribution for the superpotential was however missing there. Let us emphasize that the background is only used at infinity, it is nothing but a covariant way to impose the required boundary conditions. Our purpose will now be to use formula (4) to get the complete result.

The total superpotential will receive contributions from the  $gl(D, \mathbb{R})$  (parametrized by  $\lambda_b^a$ ) and diffeomorphism (parametrized by  $\xi^\rho$ ) gauge symmetries only<sup>8</sup>. Using the equations (10-15) in the definition (3) we obtain:

$$\mathbf{W}_{\xi, \lambda} = \mathbf{i}_\xi \theta^a \frac{\delta \mathbf{L}}{\delta \theta^a} + (\mathbf{i}_\xi \omega_b^a - \lambda_b^a) \frac{\delta \mathbf{L}}{\delta \omega_b^a} = -\mathbf{R}_b^a \wedge \mathbf{i}_\xi \Sigma_a^b - (\mathbf{i}_\xi \omega_b^a - \lambda_b^a) \mathbf{D} \Sigma_a^b \quad (23)$$

where the shorthand notation (19) was used. And  $\mathbf{i}_\xi$  denotes the interior product with the vector field  $\xi^\rho$ .

Now from equation (4), the variation of the gravitational superpotential at some boundary  $\mathcal{B}_r$  (and in particular at spatial infinity  $\mathcal{B}_\infty$ ) is given by

$$\delta \mathbf{U}_{\xi, \lambda} = \delta \omega_b^a \wedge \mathbf{i}_\xi \Sigma_a^b + (\mathbf{i}_\xi \omega_b^a - \lambda_b^a) \delta \Sigma_a^b \quad (24)$$

$$= \delta \left( (\mathbf{i}_\xi \omega_b^a - \lambda_b^a) \Sigma_a^b \right) - \mathbf{i}_\xi \left( \delta \omega_b^a \wedge \Sigma_a^b \right) \quad (25)$$

$$= \delta \left( D_b \xi^a \Sigma_a^b - \mathbf{i}_\xi (\omega_b^a \wedge \bar{\Sigma}_a^b) \right) - \mathbf{i}_\xi \left( \delta \omega_b^a \wedge \Delta \Sigma_a^b \right) \quad (26)$$

$$= \delta \left( D_b \xi^a \Sigma_a^b - \mathbf{i}_\xi \left( \Delta \omega_b^a \wedge \Sigma_a^b \right) \right) + \mathbf{i}_\xi \left( \Delta \omega_b^a \wedge \delta \Sigma_a^b \right) \quad (27)$$

The first equation (24) follows from the criterion (4) applied to (23). The second one (25) follows from the first one after some simple algebraic manipulations assuming that the variation of the gauge parameters (namely  $\xi^\rho$  and  $\lambda_b^a$ ) vanishes. The last two equations (26-27) are constructed from

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<sup>8</sup>The Projective symmetry (15) does not contribute to  $\mathbf{W}_\xi$  (see definition (3) together with (1)). This is a consequence of the fact that no gauge field is associated to this symmetry. In other words, there is no field whose  $\kappa$ -transformation law is proportional to the derivative of the gauge parameter. Hence the associated current is identically zero [1].

(25) such that the last term reproduces one of the two boundary conditions (20-21). We also used the result (17).

The next point is to integrate equation (26) or (27) using the Dirichlet or Neumann boundary condition (20-21). The equation (24) will be integrable iff

$$i_{\xi}(\delta\omega^a_b \wedge \delta\Sigma_a^b) = 0 \quad (28)$$

at spatial infinity. This integrability condition was derived apparently for the first time in [17] within the covariant symplectic formalism (the term

$$\Omega := \delta\omega^a_b \wedge \delta\Sigma_a^b \quad (29)$$

is nothing but the so-called pre-symplectic two-form). The correspondence between the covariant phase space formalism and equation (4) will be given in [5].

### **The (1/2) Komar superpotential**

Let us first use the Neumann boundary condition on the metric (Dirichlet on the connection) given by (21) and integrate (24). If we assume that  $\omega^a_b$  approaches  $\bar{\omega}^a_b$  fast enough, the last two terms of (27) vanish. Then, up to some global constant, the superpotential is given by:

$$U_{\xi}^{Ko} = D_b \xi^a \Sigma_a^b = -\frac{\sqrt{|g|}}{4\kappa^2} *(d\xi) \quad (30)$$

where  $\xi$  is the one-form associated to the vector field  $\xi^\rho$  (we also used the definition (19)).

We found the so-called Komar superpotential. However, the coefficient is not the usual one. If we compute the charge given by the above superpotential for the Schwarzschild black hole in  $D$  spacetime dimensions, we will find  $\frac{D-2}{D-3} \times M$  instead of  $M$ . The superpotential found by Komar [12] was based on another one proposed by Møller [18]. This author rescaled by hand the expression (30) to find the correct Schwarzschild mass for  $D = 4$ .

Here, we are not allowed to change the natural normalization of (30), so this superpotential is not appropriate to compute the usual conserved charges at spatial infinity. This is not surprising because it must be derived using the boundary conditions (21) which are incompatible with asymptotic flatness (or AdS).

## The KBL superpotential

Let us now use some Dirichlet boundary conditions on the metric (equation (20)) to integrate equation (24). In that case, the last term of equation (26) vanishes. Thus the superpotential is given by:

$$U_\xi^{KBL} = U_\xi^{Ko} - i_\xi(\omega^a_b \wedge \bar{\Sigma}_a^b) - C^t \quad (31)$$

Recall that equation (24) gives the superpotential up to some global constant  $C^t$ . A natural way to fix this constant is to require the vanishing of the superpotential when evaluated in the background metric  $\bar{g}^{ab}$ , that is  $U_\xi^{KBL}[\bar{g}] = 0$ . We then find a  $D$ -dimensional version of the superpotential proposed by Katz, Bičák and Lynden-Bell [11]:

$$\begin{aligned} U_\xi^{KBL} &= U_\xi^{Ko} - U_\xi^{Ko}[\bar{g}] - i_\xi(\Delta\omega^a_b \wedge \bar{\Sigma}_a^b) \\ &= U_\xi^{Ko} - U_\xi^{Ko}[\bar{g}] - i_\xi(\Delta\omega^a_b \wedge \Sigma_a^b) \end{aligned} \quad (32)$$

where equation (20) (with  $\delta\omega^a_b = \Delta\omega^a_b$ ) was again used in the second line.

It is straightforward to check that the Hodge-dual of the above  $(D-2)$ -form is in components equal to (see [1] for related calculation):

$${}^{KBL}U_\xi^{\mu\nu} = {}^{Ko}U_\xi^{\mu\nu} - {}^{Ko}\bar{U}_\xi^{\mu\nu} + S^\mu \xi^\nu - S^\nu \xi^\mu \quad (33)$$

with

$${}^{Ko}U_\xi^{\mu\nu} = \frac{\sqrt{|g|}}{4\kappa^2}(\nabla^\mu \xi^\nu - \nabla^\nu \xi^\mu) \quad \text{and} \quad S^\mu = \frac{\sqrt{|g|}}{4\kappa^2}(\Delta\Gamma_{\rho\sigma}^\mu g^{\rho\sigma} - \Delta\Gamma_{\rho\sigma}^\sigma g^{\mu\rho}). \quad (34)$$

In general, this superpotential<sup>9</sup> depends on the gauge parameter  $\xi^\rho$ , which is not arbitrary but has to be compatible with the asymptotic boundary condition [1]. In the case of Dirichlet boundary conditions (20), this parameter should be an asymptotic Killing vector, that is  $g_{\mu\nu} + \delta_\xi g_{\mu\nu} \rightarrow \bar{g}_{\mu\nu}$  as  $r \rightarrow \infty$ .

Our derivation of the KBL superpotential is independent of the space-time dimension, of the first order theory used (Palatini or Cartan-Weyl), and of the presence of a cosmological constant<sup>10</sup>. It follows straightforwardly

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<sup>9</sup>The result (33) without the background contribution was derived in [1] (and called the Katz superpotential [19]) using the cascade equations techniques.

<sup>10</sup>If we add a cosmological constant to the Hilbert Lagrangian (5) by  $L_\Lambda = \Lambda\sqrt{|g|}\Sigma$ , the calculations leading to the result (33) remain almost unchanged.

from equation (4) and from Dirichlet boundary conditions on the metric. This is to be contrasted with the more involved original derivation [11] and our derivation in [1] both with the Noether method where some prescription was needed in order to fix the ambiguity in the current  $\mathbf{J}_\xi$ . The equivalence between our present derivation and the covariant phase space method used recently in [13] will be presented in [5]. Moreover, the use of the Affine formalism for gravity gives *directly* the term proportional to  $\partial_\mu \xi^\rho$  in the KBL superpotential which was added by hand in [13] following some “covariant criterion”.

Some of the properties of the KBL superpotential, and the validity of equation (4) at null infinity are discussed in the last section.

### 3 The KBL superpotential at null infinity

As shown in the previous section, the KBL superpotential follows straightforwardly from the diffeomorphism invariance of general relativity, through equation (4) and Dirichlet boundary conditions (20). Moreover, it is the *only* superpotential which satisfies the following properties:

- It is generally covariant and then can be computed in any coordinate system.
- If the chosen coordinates are the *Cartesian* ones of an asymptotically flat (or AdS) spacetime, the KBL superpotential gives the ADM mass formula [20] (or the AD mass [21]).
- It gives the mass and angular momentum (and the Brown and Henneaux conformal charges [22] for  $AdS_3$ ) with the right normalization in any spacetime dimensions  $D \geq 3$ . More generally, it can be used for any asymptotic Killing vector  $\xi^\rho$ .

We just derived the KBL superpotential at *spatial infinity*. The important point is that it depends on some background metric  $\bar{g}_{\mu\nu}$ . This is crucial for its general covariance.

The asymptotic background metric is not a new object of spacetime. In fact, it fixes the boundary conditions, namely  $g_{\mu\nu} \rightarrow \bar{g}_{\mu\nu}$  as  $r \rightarrow \infty$ . It is imposed naturally at the boundary of spacetime. It is however clear that in general it cannot be extended arbitrarily everywhere in the bulk, for instance flat space may have a different topology than our solution sector.

The choice of such a  $\bar{g}_{\mu\nu}$  everywhere is unnatural from the background-independent Einstein theory point of view. Moreover, there are many ways to define this background metric  $\bar{g}_{\mu\nu}$  in the bulk such that its asymptotic value agrees with our chosen boundary conditions. It is not clear why one of these choices would be better than the others. If we cannot properly define  $\bar{g}_{\mu\nu}$  *everywhere* then we will not be able to define  ${}^{KBL}U_{\xi}^{\mu\nu}$  in the bulk and interpret the KBL superpotential as an expression for a quasi-local mass. We believe the problem of quasi-local charges is ill posed and needs to be supplemented by specific boundary assumptions at the surface to be used for enveloping the physical object.

Suppose now that our spacetime is also flat at future (past) null infinity  $I^+$  ( $I^-$ ). In that case, we can uniquely extend the background metric  $\bar{g}_{\mu\nu}$  to that region. The KBL superpotential can then be covariantly defined also on  $I^+$ . In particular, we can integrate equation (2) on a piece of  $I^+$ , namely  $\Delta$ , bounded by spatial infinity  $i_0$  and some time-dependent cross section  $\mathcal{C}$  [9]. The on-shell result is

$$\int_{\Sigma_{\infty}} \mathbf{U}_{\xi}^{KBL} = \int_{\mathcal{C}} \mathbf{U}_{\xi}^{KBL} + \int_{\Delta} J_{\xi}^{KBL} \quad (35)$$

with  $J_{\xi}^{KBL} = d\mathbf{U}_{\xi}^{KBL}$  (on-shell).

It has been proved by Katz and Lerer [23] that the first term in the rhs of (35) reproduces the Bondi mass [24], the Penrose linear momentum [25] and the Penrose-Dray-Streubel [26] angular momentum at null infinity (depending on the choice of the asymptotic  $\xi^{\rho}$ ). The second integral in the rhs of (35) is then nothing but the total amount of charge which crossed  $\Delta$ . In the case where  $\xi^{\rho}$  is an asymptotic BMS translation [27], we recover the result pointed out in [9].

As will be shown in the forecoming paper [5], the equation (4) used to compute the superpotential is equivalent to the covariant symplectic phase space methods [8, 9, 3]. In particular, it was shown by Wald and Zoupas [17] that equation (4) is non integrable at null infinity. This is due in part to the fact that the use of equation (4) is justified only if the charge is conserved [4], that is, if the flux of the Noether current vanishes on the spacetime boundary  $\mathcal{H}_r$  under consideration. This is not the case in general for the Bondi-type charges. The point is that the equation for the variation of the superpotential can be derived from the vanishing of the flux at infinity provided the set of boundary conditions is “Lagrangian” ie provided it leads to the vanishing of the symplectic form (29) there. We will come back to these statements in [5].

As a final comment, it seems quite natural that a formula like (35) should also exist on the horizon  $\mathcal{H}$  of a black hole. More precisely, if we consider  $\mathcal{H}$  as a boundary of spacetime, with some boundary conditions on it, we would hope to find an associated superpotential, and then, some conserved or Bondi-type charges depending on its dynamical behavior. The work of Ashtekar et al. [28] goes precisely in that direction. So at least for isolated horizons the definition of a quasi-local mass on the horizon (even in the nonstationary case) should go through beyond the four dimensional case they have considered.

**Acknowledgments.** We are grateful to AEI for hospitality, and to A. Ashtekar, J. Bičák and M. Henneaux for discussions.

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